

Neutrino Dispersion in Magnetized Media and Spin Oscillations in the Early Universe

Per Elmfors

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Dario Grasso

*Department of Theoretical Physics, Uppsala University,
Box 803, S-751 08 Uppsala, Sweden,
and Department of Physics, University of Stockholm,
Vanadisvägen 9, S-113 46 Stockholm, Sweden*

Georg Raffelt

*Max-Planck-Institut für Physik, Föhringer Ring 6,
D-80805 Munich, Germany*

Abstract

We derive general expressions for the neutrino dispersion relation in a magnetized plasma with a wide range of temperatures, chemical potentials, and magnetic field strengths. If the electron and proton chemical potentials vanish, as in the early Universe, there is no magnetization contribution to the neutrino refractive index to leading order in the Fermi coupling constant, contrary to claims in the recent literature. Therefore, as long as the magnetic field satisfies $B \lesssim T^2$, the neutrino refractive index in the early Universe is dominated by the standard “non-local term”. If neutrinos are Dirac particles with magnetic moment μ , then their right-handed components are thermally populated before the nucleosynthesis epoch by magnetically induced spin oscillations if $\mu B_0 \gtrsim 10^{-6} \mu_B$ gauss, where $\mu_B = e/2m_e$ is the Bohr magneton and B_0 is a large-scale primordial magnetic field at $T_0 \approx 1$ MeV. For a typically expected random field distribution, even smaller values for μB_0 would suffice to thermalize the right-handed Dirac components.

I. INTRODUCTION

If neutrinos carry magnetic or electric dipole or transition moments, they can spin-precess into other spin and/or flavour states in the presence of external magnetic fields. For example, if neutrinos were Dirac particles with a magnetic dipole moment μ , the active left-handed states could spin-precess into the otherwise sterile right-handed ones. It has been speculated that this effect can explain the deficiency of the measured solar neutrino fluxes, and it certainly can be important for supernova physics where large magnetic fields are known to exist [1]. Further, it has been recognized for a long time that primordial magnetic fields of sufficient strength would couple right-handed Dirac neutrinos to the cosmic thermal heat bath and thus cause these “wrong-helicity” states to be thermally populated [2]. This effect would enhance the expansion rate of the Universe at the epoch of nucleosynthesis and thus modify the standard scenario of the formation of the light elements, in potential disagreement with the observationally inferred abundances.

The original discussions of this cosmological effect [2] did not take into account neutrino dispersion, which at that time had received only marginal attention. Later on, it became clear that even though the neutrino dispersion relations in vacuum and in media are very close to that of massless particles, any deviation from the latter may cause significant modifications of spin or flavour-oscillation processes. A first assessment of medium-induced dispersion effects for early-Universe magnetic spin oscillations was provided in Ref. [3]. In addition, however, one has to worry about neutrino collisions during the oscillation process. A formalism for the simultaneous treatment of oscillations and collisions was pioneered in Refs. [4,5], and was refined in terms of quantum-kinetic equations in Refs. [6]. A quantum-kinetic treatment of the early-Universe magnetic oscillation problem was provided in a recent series of papers [7–10].

Because even fine points of the neutrino dispersion relation are important for oscillation phenomena, one naturally wonders if the assumed presence of a strong magnetic field may cause a spin polarization of the electrons and positron in the medium, which in turn may act as a new contribution to the dispersion relation. Semikoz and Valle [8] claim that this is the case even for zero chemical potential, and that this effect dominates the neutrino dispersion relation for the physical conditions relevant in the early Universe.

Upon closer inspection, however, we find that this dispersion relation is based on an unfortunate sign error. In a charge-symmetric plasma, the magnetization part of the local self-energy terms cancels between electrons and positrons rather than adding, as claimed by Semikoz and Valle [8]. While the correct sign can be understood by a simple physical argument (Sect. II B 2) and from the requirement of CPT invariance (Sect. II B 3), we take this opportunity to provide the neutrino dispersion relation in a magnetized medium for arbitrary electron chemical potential and magnetic field strength. The correct sign is then a consequence of our completely general and formal derivation, which leaves no room for ambiguities. Our general expressions may also be of interest in the context of neutrino spin oscillations in supernovae, where strong fields and very degenerate electrons occur. Surprisingly,

we find that even for arbitrary field strengths our expressions are very similar to those derived by D’Olivo, Nieves, and Pal [13] in the weak-field limit.

Neutrino dispersion in a magnetized medium may be viewed from a somewhat different perspective where one considers an effective neutrino electromagnetic form factor, or vertex function, induced by the presence of the medium [13–16]. Various components of this vertex function, which is a Lorentz tensor, may be interpreted as certain effective neutrino electromagnetic multipole moments. In this language, neutrino dispersion in a magnetized medium is represented by a medium-induced effective neutrino magnetic dipole moment, which naturally leads to an energy shift in the presence of a magnetic field.¹ The results of Refs. [13,14] imply that in a charge-symmetric plasma this dipole moment vanishes, in agreement with our present calculations and arguments. The same conclusion was reached in an early paper by Semikoz [15], in conflict with the later finding of Semikoz and Valle [8].

In Sect. II we derive general expressions for the neutrino dispersion relation in a magnetized medium, and we derive the relative sign of the magnetization effect by a direct physical argument. In Sect. III we investigate the efficiency of primordial neutrino spin oscillations in view of the correct neutrino dispersion relation in a magnetized plasma which, in the early Universe, is well approximated by the dispersion relation of an unmagnetized medium. Section IV is devoted to a summary and discussion.

II. NEUTRINO DISPERSION IN MAGNETIZED MEDIA

A. General Self-Energy Diagrams

In order to derive a general expression for the neutrino dispersion relation in a magnetized medium we observe that, to lowest order, the self-energy is given by the tadpole and bubble diagrams shown in Fig. 1. To be specific we shall derive the dispersion relation for electron neutrinos; more general cases can be inferred by simple substitutions.

The neutrino self-energy contribution from the tadpole diagram with an arbitrary fermion loop is

$$-i\Sigma_{\text{tadpole}} = -\frac{1}{4} \left(\frac{ig}{\cos\theta_W} \right)^2 \text{tr} [\gamma^\alpha (c_V - c_A\gamma_5) iS(x, x)] iD_{\alpha\beta}^Z(0) \gamma^\beta L, \quad (1)$$

where g is the weak gauge-coupling constant and θ_W the weak mixing angle. We use the notation $R \equiv \frac{1}{2}(1+\gamma_5)$ and $L \equiv \frac{1}{2}(1-\gamma_5)$. Further, $D_{\alpha\beta}^Z(\Delta)$ is the Z -boson propagator while $S(x, y)$ is the coordinate-space propagator for the background fermion.

¹The use of an “effective magnetic dipole moment” to describe the neutrino energy shift in a magnetized medium is somewhat misleading, because the γ -structure of the vertex function is not that of a magnetic dipole interaction. Among other differences, only left-handed states experience any shift at all.

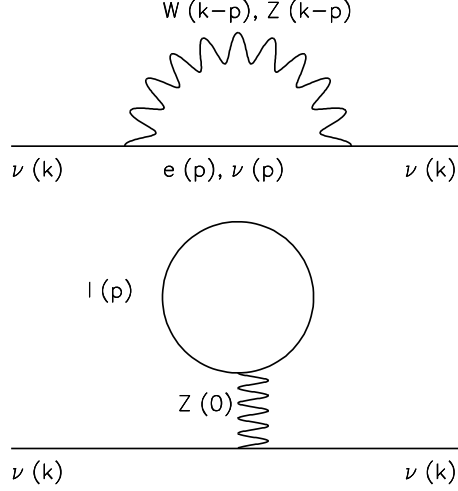


FIG. 1. The tadpole and the bubble diagrams.

For a charged Dirac spin- $\frac{1}{2}$ particle in the presence of an external magnetic field, $S(x, x)$ is given in Appendix A. Our prime example is electrons for which the weak coupling constants are $c_V = -\frac{1}{2} + 2\sin^2\theta_W$ and $c_A = -\frac{1}{2}$.

The bubble diagram contributes only for a background of charged leptons from the same family as the test neutrino. In our specific case of a test ν_e in the presence of an e^+e^- plasma, we find

$$-i\Sigma_{\text{bubble}} = \left(\frac{ig}{\sqrt{2}}\right)^2 R \int \frac{d^4p}{(2\pi)^4} \gamma^\alpha iS(p) iD_{\alpha\beta}^W(k-p) \gamma^\beta L . \quad (2)$$

For a neutrino background from the same family as the test neutrino, there is a similar diagram with a Z - ν -loop that can be obtained by replacing $g^2 \rightarrow g^2/2\cos^2\theta_W$ and $m_W \rightarrow m_Z$.

The tadpole diagram provides only a local contribution, i.e. the gauge-boson propagator is taken at the energy-momentum transfer $\Delta = 0$ so that we could have used an effective low-energy four-fermion interaction. The bubble diagram, however, involves the gauge-boson propagator at a non-vanishing Δ so that there is a non-local term in the self-energy. Even in extreme astrophysical sites, such as neutron stars, the relevant energies are so low, and the chemical potential so high relative to the temperature, that the bubble diagram is dominated by the local term. However, the local term vanishes identically in a charge-symmetric plasma. Therefore, in the early Universe the neutrino self-energy is dominated by the non-local part of the bubble diagram [11].

B. Local Terms

1. Formal Derivation

In order to derive the electron-neutrino dispersion relation in a magnetized medium explicitly, we begin with the local contributions. To this end we expand the gauge-boson propagators in powers of the energy-momentum transfer Δ ,

$$D_{\alpha\beta}^{W,Z}(\Delta) = \frac{g_{\alpha\beta}}{m_{W,Z}^2} + \frac{g_{\alpha\beta}\Delta^2 - \Delta_\alpha\Delta_\beta}{m_{W,Z}^4} + \mathcal{O}\left(\frac{\Delta^4}{m_{W,Z}^6}\right) . \quad (3)$$

The first term, which is the only one contributing to the tadpole, gives the local part of the self-energy.

Using the charged-fermion propagator in an external magnetic field (see Appendix A for more details), the tadpole yields for a plasma consisting of electrons, neutrinos, and nucleons:

$$\Sigma_{\text{tadpole}} = \frac{G_F}{\sqrt{2}} \left\{ \left[-N_{n-\bar{n}} + 2 \sum_{i=e,\mu,\tau} N_{\nu_i-\bar{\nu}_i}^L - (1 - 4 \sin^2 \theta_W)(N_{e-\bar{e}} - N_{p-\bar{p}}) \right] \gamma_0 + N_{e-\bar{e}}^0 \hat{\mathbf{B}} \cdot \boldsymbol{\gamma} \right\} L , \quad (4)$$

where $\hat{\mathbf{B}}$ is a unit vector in the external B -field direction. Further, $N_{f-\bar{f}}$ denotes the net number density of fermions f , i.e. the total number density of fermions f minus that of antifermions \bar{f} . For neutrinos, only the number density of left-handed states (superscript L) is counted, which is identical to the total number density unless the right-handed degrees of freedom have been populated by, say, magnetically induced spin oscillations. Usually, the standard electron and proton terms cancel against each other in a charge-neutral plasma where $N_{e-\bar{e}} - N_{p-\bar{p}} = 0$.

In the magnetic tadpole term, $N_{e-\bar{e}}^0$ is the net number density of electrons in the lowest Landau level. Of course, the exact cancellation of all higher Landau levels applies only to Dirac fermions which do not carry anomalous magnetic dipole moments. This approximation is not justified for nucleons, which carry large anomalous magnetic moments so that their polarization does not cancel between the higher Landau levels which are not degenerate. However, unless the field is extremely strong or the temperature much higher than the nucleon masses, the nucleon magnetization is suppressed by their heavier masses relative to electrons. Because in the present paper we are primarily interested in early-Universe physics between the QCD phase transition and Big-Bang nucleosynthesis (BBN), nucleons can certainly be ignored with regard to neutrino dispersion effects.

In addition we need to consider the bubble diagram, which yields a local contribution from electrons and electron neutrinos of

$$\Sigma_{\text{bubble}} = \frac{G_F}{\sqrt{2}} 2 \left[(N_{\nu_e-\bar{\nu}_e}^L + N_{e-\bar{e}}) \gamma_0 - N_{e-\bar{e}}^0 \hat{\mathbf{B}} \cdot \boldsymbol{\gamma} \right] L . \quad (5)$$

Nucleons never contribute to this term.

In the weak-field limit our results agree with those found in Ref. [13], except for the overall sign which is related to the convention in Ref. [13] that $e < 0$ for electrons.

The approach in Ref. [13] was strictly perturbative in that a plane-wave basis for the fermions was used instead of Landau levels. We stress that exact expressions for quantities such as the magnetization or the magnetic susceptibility do not in general admit a power-series expansion in B , forcing one to use Landau levels as external states [19]. However, when a quantity does admit a power series expansion, it is not too surprising that the linear term of the exact result agrees with a perturbative calculation based on plane-wave states.

Our magnetic neutrino self-energy terms apply for $B \ll m_W^2$, but B may well be large compared with other scales in the problem, such as the electron mass or the temperature. Even for such large fields the linear term actually gives the complete result. This surprising finding is traced to the fact that only the lowest Landau level contributes and that $N_{e-\bar{e}}^0$ is strictly linear in B . It must be noted, however, that the presence of the field affects the phase-space distribution of the charged fermions and thus the relationship between chemical potential and density. Therefore, one must specify if the charged-particle densities or their chemical potentials are held fixed in order to specify the functional dependence of the neutrino dispersion relation on B .

The dispersion relation for left-handed electron neutrinos in a magnetized plasma is obtained by taking the determinant of $\gamma k - \Sigma_{\text{tadpole}} - \Sigma_{\text{bubble}}$. We find

$$E_{\pm} = \pm k_0 = \pm a + |\mathbf{k} - \mathbf{b}| \quad , \quad (6)$$

where \pm refers to ν_e and $\bar{\nu}_e$, respectively. Further,

$$\begin{aligned} \frac{a}{\sqrt{2} G_F} &= -\frac{1}{2} N_{n-\bar{n}} + \sum_{i=e,\mu,\tau} N_{\nu_i-\bar{\nu}_i}^L + N_{\nu_e-\bar{\nu}_e}^L + N_{e-\bar{e}} \\ &\quad - (\frac{1}{2} - 2 \sin^2 \theta_W) (N_{e-\bar{e}} - N_{p-\bar{p}}) \quad , \\ \frac{\mathbf{b}}{\sqrt{2} G_F} &= \frac{1}{2} N_{e-\bar{e}}^0 \hat{\mathbf{B}} \quad . \end{aligned} \quad (7)$$

It is the medium- and field-induced breaking of Lorentz invariance that generates a non-trivial dispersion relation, or refractive index, for neutrino propagation. In a charge-neutral plasma the term proportional to $(N_{e-\bar{e}} - N_{p-\bar{p}})$ vanishes. Again, there is a small nucleon contribution to \mathbf{b} which we have neglected. We stress that it is a slight abuse of language to call \mathbf{b} magnetization because only the spin part of the magnetization enters, not the orbital part. Note further that the spin is not a conserved quantity and only the lowest Landau level is a spin eigenstate.

2. Physical Derivation

Because the local magnetization contribution to the refractive index is controversial in the literature, it is useful to provide a more physical derivation where the absolute sign, and the relative sign between the electron and positron terms, become more directly apparent. To this end we may start directly from the four-fermion neutrino vertex with a charged lepton ℓ :

$$\mathcal{H}_{\text{int}} = \sqrt{2} G_F \bar{\Psi}_{\nu} \gamma_{\alpha} L \Psi_{\nu} \bar{\Psi}_{\ell} \gamma^{\alpha} (g_V - g_A \gamma_5) \Psi_{\ell} \quad . \quad (8)$$

Here, the effective weak neutral-current coupling constants $g_{V,A}$ are identical with $c_{V,A}$ unless ℓ is from the same family as the neutrino, in which case $g_{V,A} = c_{V,A} + 1$ because the Fierz-transformed charged-current mimics a neutral-current interaction.

The neutrino self-energy is found by calculating the expectation value $\langle \bar{\Psi}_\ell \gamma^\alpha (g_V - g_A \gamma_5) \Psi_\ell \rangle$ in a background bath of fermions ℓ . In an unpolarized, isotropic medium only the zeroth component of the vector current contributes and yields the standard result. A magnetically induced polarization of the charged background fermions, however, causes the axial current to obtain a non-vanishing expectation value.

For ultrarelativistic charged fermions the expectation value of the chirality operator γ_5 is identical with that of $\text{sign}(q) \hat{\mathbf{p}} \cdot \hat{\mathbf{B}} \lambda$, an observation that establishes a simple relation between chirality and the magnetic quantum number λ of the Landau levels. It implies that the axial-vector contributions cancel between charged fermions with the same momentum but opposite λ . The Landau-level energies $E_{n,\lambda,p_z}^2 = m^2 + p_z^2 + |qB|(2n + 1 - \lambda)$, with $n = 0, 1, 2, \dots$ and $\lambda = \pm 1$, are degenerate between the levels $(n, \lambda = +1)$ and $(n - 1, \lambda = -1)$ except for the lowest level $(n = 0, \lambda = +1)$, which is not matched by a lower level with opposite magnetic quantum number. Therefore, only the lowest Landau level contributes to the expectation value of the axial-vector current.

A negatively charged ultrarelativistic ℓ in the lowest Landau level, moving along the B -field, has its magnetic moment parallel to \mathbf{B} , a spin opposite to \mathbf{B} , and therefore negative chirality. Hence for such a state

$$\langle \bar{\Psi}_\ell \gamma (g_V - g_A \gamma_5) \Psi_\ell \rangle = \langle \bar{\Psi}_\ell \boldsymbol{\gamma} \Psi_\ell \rangle (g_V + g_A) = \hat{\mathbf{B}} (g_V + g_A) . \quad (9)$$

If ℓ moves in the opposite direction we get $-\hat{\mathbf{B}}(g_V - g_A)$ so that the vector part averages to zero for each momentum mode separately if the phase-space distribution is reflection-symmetric along \mathbf{B} . An anti- ℓ ($\bar{\ell}$) moving along the B -field has its magnetic moment also pointing parallel to \mathbf{B} , but its spin in the opposite direction relative to an ℓ with the same momentum along the field, since the charge is opposite. Therefore, the helicity of $\bar{\ell}$ is opposite to that of ℓ and thus their chiralities are equal. The expectation value corresponding to Eq. (9) is then $-\hat{\mathbf{B}}(g_V + g_A)$. Similarly we get $\hat{\mathbf{B}}(g_V - g_A)$ for an $\bar{\ell}$ moving in the opposite direction. Multiplying with the net number density of ℓ 's and $\bar{\ell}$'s in the lowest Landau level we obtain

$$\Sigma_\nu = -g_A \sqrt{2} G_F N_{\ell-\bar{\ell}}^0 \hat{\mathbf{B}} \cdot \boldsymbol{\gamma} L , \quad (10)$$

where the final minus sign comes from the contraction of space-like indices in Eq. (8). While our simple derivation was based on the notion of ultrarelativistic charged leptons, this result holds true even for non-relativistic ones as follows from the formal derivation in the previous section.

The absolute sign of the energy shift, which differs from the one found in Ref. [13], can be checked by comparing Eqs. (4) and (5) with Eq. (10). If the leptons are of a family other than the neutrinos, $g_{V,A} = c_{V,A}$, and one must compare Eq. (10) with the tadpole term alone. For leptons of the same family, $g_{V,A} = c_{V,A} + 1$, and the sum of the bubble and the tadpole diagram should be compared to Eq. (10). The relative sign between the electron and positron terms also follows directly from this simple derivation without ambiguity.

In the early Universe, the numbers of particles and antiparticles are believed to be identical to within about 10^{-9} . Therefore, the plasma was effectively charge-symmetric. Our general results, Eqs. (4) and (5), reveal that, to leading order in m_W^{-2} , there is no magnetization contribution to the neutrino refractive index in such an environment, contrary to what has been claimed by Semikoz and Valle [8] who found that the fermion and antifermion terms in Eqs. (4) and (5) add rather than subtract. It is correct as in Eq. (2.9) of Ref. [8] to identify the relevant spatial part of the axial current $\langle \bar{\Psi}_e \gamma \gamma_5 \Psi_e \rangle$ with the difference between the electron and positron magnetizations, but in the manipulations leading from their Eq. (3.3) to (3.6) Semikoz and Valle have unfortunately picked up an incorrect minus sign. Therefore, at epochs before nucleosynthesis the non-local neutrino refractive terms remain more significant than the local ones [11], even in the presence of strong magnetic fields.

3. CPT Argument

The vanishing of the local contribution to the neutrino self-energy in a CP-symmetric plasma can also be deduced from a direct symmetry argument. To this end we assume that the background plasma is CP symmetric, and in addition we assume that it is in a stationary state so that it is also symmetric under the time-reversal operation T. Since CPT is strictly conserved in our theory, and the magnetic field is CPT invariant, it follows that neutrinos and antineutrinos of a given momentum must experience the same medium-induced energy shift, i.e. their self-energy in the medium must be the same. Put another way, the expectation value of $\bar{\Psi}_\nu \Sigma \Psi_\nu$ must be the same for neutrino and antineutrino states of equal momenta.

At one-loop level, the general form of the self-energy operator Σ in a magnetized medium is [13]

$$\Sigma = R(ak^\mu + bu^\mu + c\tilde{B}^\mu)\gamma_\mu L \quad . \quad (11)$$

Here, k is the four-momentum of the test (anti)neutrino, u is the four-velocity of the background medium, and $\tilde{B}_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$ is a covariant expression for the external electromagnetic field which is a pure B-field in the rest frame of the medium. The coefficients a , b , and c are functions of the scalars k^2 , $\omega \equiv k \cdot u$, and $k \cdot \tilde{B}$.

Under CPT the current $\bar{\Psi}_\nu \gamma^\mu \Psi_\nu$ and the four-momentum k change sign. (Recall that the Dirac eq. implies $\bar{\Psi}_\nu k^\mu \gamma_\mu \Psi_\nu = m \bar{\Psi}_\nu \Psi_\nu$, and that $\bar{\Psi}_\nu \Psi_\nu$ is invariant under CPT.) However, the four-vectors u and \tilde{B} are invariant under CPT. It is important to observe here that u is not an operator for Ψ_ν since it is just fixing the new reference frame. From Eqs. (1) and (2) it is clear that the local contribution to Σ is independent of k so that the coefficient a must be zero while b and c must be constants. However, because u^μ and \tilde{B}^μ are even under CPT while $\langle \bar{\Psi}_\nu \gamma_\mu \Psi_\nu \rangle$ is odd, and because $\langle \bar{\Psi}_\nu \Sigma \Psi_\nu \rangle$ is required to be even, we find that b and c must be zero. The coefficient c is related to a medium-induced effective neutrino magnetic dipole moment. In Ref. [20] it was already shown on the basis of the same argument that such a dipole moment must vanish. Thanks to this argument no contributions

to Σ_{local} can arise from strong-field corrections to the W propagator in a CPT symmetric plasma. Non-local terms which are odd functions of scalars that are odd under CPT, namely ω and $k \cdot \tilde{B}$, are not required to vanish.

C. Non-local Terms

In a charge-symmetric plasma, all of the local self-energy terms given in Eq. (7) vanish so that the second term in the expansion of the gauge-boson propagator in Eq. (3) dominates. We shall concentrate on the case where $m_e \ll T \ll m_W$ and $B \lesssim T^2$. In the early Universe, these are quite reasonable approximations between the QCD phase transition and nucleosynthesis. Repeating the calculations for the bubble diagram we then obtain

$$\begin{aligned} \Sigma_{\text{bubble}}(k) = & -\frac{7\sqrt{2}\pi^2 G_F T^4}{45 m_Z^2} \left(1 + \frac{2m_Z^2}{m_W^2}\right) \left(\gamma_0 k_0 - \frac{1}{4}\gamma k\right) L \\ & -\frac{\sqrt{2} G_F T^2}{6 m_W^2} e\mathbf{B} \cdot \boldsymbol{\sigma} \left[\gamma_0 k_0 + (\hat{\mathbf{B}} \cdot \boldsymbol{\gamma})(\hat{\mathbf{B}} \cdot \mathbf{k})\right] L \quad , \end{aligned} \quad (12)$$

where $\boldsymbol{\sigma}$ is a vector of Dirac spin matrices defined by $\mathbf{B} \cdot \boldsymbol{\sigma} = \frac{i}{4}F^{\mu\nu}[\gamma_\mu, \gamma_\nu]$, where $F^{\mu\nu}$ is the field strength tensor. The resulting dispersion relation takes the form

$$\begin{aligned} E_{\pm} = \pm k_0 = & \left[1 - \frac{7\sqrt{2}\pi^2 G_F T^4}{45 m_Z^2} \left(1 + \frac{2m_Z^2}{m_W^2}\right)\right] |\mathbf{k}| \pm \frac{\sqrt{2} G_F T^2}{3 m_W^2} e\mathbf{B} \cdot \mathbf{k} \\ \approx & \left(1 - 6.0 \frac{G_F T^4}{m_W^2}\right) |\mathbf{k}| \pm 0.47 \frac{G_F T^2}{m_W^2} e\mathbf{B} \cdot \mathbf{k} \quad . \end{aligned} \quad (13)$$

The first part agrees with the result of Ref. [11]; it is the same for ν_e and $\bar{\nu}_e$. The B -dependent energy shift is anisotropic and opposite for ν_e and $\bar{\nu}_e$. However, it remains subdominant compared to the isotropic term as long as $B \lesssim T^2$.

III. SPIN OSCILLATION IN THE EARLY UNIVERSE

A. Neutrino Depolarization Rate

As an application of our results we consider explicitly the case of Dirac neutrinos with a magnetic moment μ . In the presence of a primordial magnetic field the thermally populated left-handed (l.h.) states can spin-precess into the otherwise sterile right-handed (r.h.) ones, thus which will be populated as well. This process of populating the “wrong-helicity” neutrino states is treated theoretically by virtue of a Boltzmann-type kinetic equation, which includes neutrino oscillations as discussed in Refs. [6]. However, for a simple estimate one may ignore the detailed evolution of the individual momentum modes and rather study an average evolution of the overall spin-polarization vector \mathbf{P} of the entire ensemble. In this simplified approach the global spin-polarization vector evolves as [5]

$$\partial_t \mathbf{P} = \mathbf{V} \times \mathbf{P} - D \mathbf{P}_T, \quad (14)$$

where \mathbf{V} is a vector of effective magnetic interaction energies, D a damping rate due to collisions, and \mathbf{P}_T the “transverse” part of the spin-polarization vector to be discussed below.

In the absence of a medium, the damping rate vanishes and the effective interaction energy for ultrarelativistic neutrinos is $\mathbf{V} = 2\mu\mathbf{B}_T$, where \mathbf{B}_T is the component of the \mathbf{B} field which is transverse to the neutrino direction of motion [1,12]. Because only the transverse magnetic field matters, in vacuum the neutrino helicity can be reversed entirely by spin precessions. Put another way, l.h. and r.h. states are maximally mixed by the presence of a magnetic field, independently of the field direction with respect to the neutrino direction of motion, unless \mathbf{B}_T vanishes exactly. Of course, the precession time depends on the magnitude of \mathbf{B}_T and thus on the relative field direction.

The first impact of a medium is that it endows the active (l.h.) neutrino states with a nontrivial dispersion relation, while the sterile (r.h.) ones remain unaffected. Because the particle-antiparticle asymmetry in the early Universe is thought to be of order 10^{-9} for all species, i.e. small, the dominant contribution to the neutrino refractive index is the non-local term that was first identified in Ref. [11]. The discussion in Sect. IIC reveals that even in a magnetized charge-symmetric plasma, the additional neutrino refractive term from the B field is rather small so that the standard isotropic term continues to dominate, in agreement with the treatment of Ref. [3]. We expect this to remain true, and that the first term of Eq. (12) remains good as an approximation even for temperatures not much higher than the electron mass which are relevant at the time of the BBN. However, in Ref. [3] the impact of the damping term was not properly discussed. The later systematic studies of kinetic equations for oscillating neutrinos were not available at that time.

In order to identify \mathbf{V} and D relevant for the conditions of the early Universe we begin with the energy difference between l.h. and r.h. neutrinos of flavour $\ell = e, \mu$, or τ which is $E_{\text{l.h.}} - E_{\text{r.h.}} = -\xi E$. Here, E is the unperturbed energy, which agrees with $E_{\text{r.h.}}$ because r.h. neutrinos do not experience any energy shift in the medium. Assuming the mass of the neutrino to be much smaller than the temperature, it is easy to extract the coefficient ξ from Eq. (13):

$$\xi = \frac{8\sqrt{2}G_F}{3} \left(\frac{\rho_{\nu_\ell + \bar{\nu}_\ell}^L}{m_Z^2} + \frac{\rho_{\ell + \bar{\ell}}}{m_W^2} \right). \quad (15)$$

Here $\rho_{\nu_\ell + \bar{\nu}_\ell}^L$ is the energy density in l.h. neutrinos plus antineutrinos of flavour ℓ while $\rho_{\ell + \bar{\ell}}$ is the energy density in the ℓ -flavoured charged leptons plus antileptons. The coefficient ξ has the same sign for neutrinos and antineutrinos as test particles. For τ neutrinos in the early Universe it is dominated by the $\rho_{\nu_\tau + \bar{\nu}_\tau}^L$ term because the presence of τ leptons is suppressed by a Boltzmann factor $e^{-m_\tau/T}$.

In a magnetized charge-symmetric plasma the spin-polarization vector of an ultrarelativistic neutrino or antineutrino of energy E evolves according to Eq. (14) with

$$\begin{aligned}\mathbf{V}_T &= 2\mu\mathbf{B}_T \quad , \\ |\mathbf{V}_L| &= \xi E \quad ,\end{aligned}\tag{16}$$

where T and L are understood to be transverse and longitudinal relative to the neutrino direction of motion. Since we are using neutrino helicity states, the direction of spin-quantization is identical with the direction of motion. Therefore, the tilt of \mathbf{V} relative to the direction of motion is twice the effective in-medium mixing angle [5] between l.h. and r.h. states:

$$\tan 2\theta = \frac{V_T}{V_L} = \frac{2\mu B_T}{\xi E} \quad ,\tag{17}$$

where $V_{T,L} = |\mathbf{V}_{T,L}|$ and $B_T = |\mathbf{B}_T|$. Thus, for sufficiently weak magnetic fields the l.h. and r.h. states are effectively de-mixed so that l.h. states spin-precess only partially into r.h. ones. Put another way, the spin precession is about the direction of an effective magnetic field $\mathbf{B}_{\text{eff}} \equiv \mathbf{V}/\mu$ which is no longer transverse to the direction of motion.

The second effect of a medium is that l.h. neutrinos scatter, thereby interrupting the precession process. A collision essentially amounts to a “measurement” of the helicity content of a given neutrino because the l.h. component is scattered out of its previous direction of propagation while the r.h. component moves on unscathed. This implies that every collision resets the neutrino into a helicity eigenstate and the oscillation process begins from scratch. Collisions thus destroy the phase coherence between the l.h. and r.h. component of a neutrino state, which amounts to a damping of the transverse part \mathbf{P}_T of the polarization vector. In the present situation where the r.h. component does not interact at all, the damping rate D is found to be *half* the collision rate of the l.h. component [5,6] so that, in the early Universe:

$$D = f_D \frac{7\pi}{48} G_F^2 T^4 E \quad .\tag{18}$$

Here, f_D is a numerical factor, which was found to be unity for μ - or τ -flavoured (anti)neutrinos in a background medium of e^\pm and all sequential (anti)neutrinos [11]. Corrections from the magnetic field are expected to be small for $eB \lesssim T^2$.

For an estimate of the rate of depolarization Γ_{depol} of the initially l.h. (anti)neutrino population, we turn to thermal averages of the refractive and damping terms. The average energy is $\langle E \rangle \approx 3T$ for a given neutrino species where the equality would be exact if the neutrinos would follow a Maxwell-Boltzmann distribution instead of a Fermi-Dirac one. Then

$$\langle D \rangle = f_D \frac{7\pi}{16} G_F^2 T^5\tag{19}$$

is the average damping term. For the refraction term in Eq. (15) we note that the energy density in one flavour ν_ℓ of l.h. (anti)neutrinos is $\rho_{\nu_\ell+\bar{\nu}_\ell} = (7\pi^2/120) T^4$. Further, $m_Z^{-2} = (\sqrt{2} G_F/\pi\alpha) \sin^2 \theta_W \cos^2 \theta_W$ where $\alpha \approx 1/137$ is the fine-structure constant. We will always approximate $\sin^2 \theta_W = 1/4$. With $\langle E \rangle = 3T$ we thus find

$$\langle V_L \rangle \approx f_L \frac{7\pi}{40\alpha} G_F^2 T^5 \quad ,\tag{20}$$

where $f_L \equiv 1 + (\rho_{\ell+\bar{\ell}}/\rho_{\nu_\ell+\bar{\nu}_\ell}^L)(m_Z/m_W)^2$ is a factor to account for the possible presence of charged leptons of flavour ℓ , which would also contribute to Eq. (15). For μ and τ neutrinos the lepton densities are small and we have $f_L = 1$, while for e neutrinos $f_L \approx 3.6$.

These results are enough to determine that the evolution Eq. (14) of the neutrino polarization vector is weakly damped, i.e. that it typically precesses several times between collisions. The oscillation or spin-precession frequency is identical to $V = (\mathbf{V}_T^2 + \mathbf{V}_L^2)^{1/2} > V_L$. With Eqs. (19) and (20) we find

$$\frac{\langle D \rangle}{\langle V_L \rangle} \approx \frac{f_D}{f_L} \frac{5\alpha}{2} . \quad (21)$$

The average period of spin precession is approximately $2\pi/\langle V_L \rangle$ so that there are at least about 10 revolutions between collisions.

In order to understand the solution of Eq. (14) in the weak-damping limit we multiply both sides with \mathbf{P}/P^2 , which leads to $\partial_t P/P = -D (P_T/P)^2$ with $P = |\mathbf{P}|$ and $P_T = |\mathbf{P}_T|$. In the weak-damping limit we may use a precession-averaged \overline{P}_T , which is found by taking the transverse part of the projection of \mathbf{P} on \mathbf{V} . Elementary geometry yields $\overline{P}_T/P = \cos 2\theta \sin 2\theta$ so that in the weak-damping limit

$$\partial_t P/P = -D \cos^2 2\theta \sin^2 2\theta , \quad (22)$$

where the effective mixing angle is given by Eq. (17). Assuming that it is small, so that $\cos 2\theta \approx 1$ and $\sin 2\theta \approx \tan 2\theta$ we thus find

$$\Gamma_{\text{depol}} \approx \frac{(2\mu B_T)^2 \langle D \rangle}{\langle V_L^2 \rangle} \approx \frac{f_D}{f_L^2} \frac{400 \alpha^2}{7\pi} \frac{\mu^2 B_T^2}{G_F^2 T^5} \quad (23)$$

for the average depolarization rate $\Gamma_{\text{depol}} \equiv \langle D \cos^2 2\theta \sin^2 2\theta \rangle$. We have freely factorized the thermal averaging process, and we have assumed a homogeneous magnetic field.

B. Comparison with Expansion Rate

In order to decide whether the depolarization of the initially l.h. neutrino ensemble is ever complete during the cosmic evolution, we need to compare Γ_{depol} with the expansion rate H . If at some epoch $\Gamma_{\text{depol}} \gtrsim H$ then r.h. neutrinos have approximately reached thermal equilibrium at that time. If this epoch falls between the QCD phase transition at $T_{\text{QCD}} \approx 150 \text{ MeV}$ and nucleosynthesis at $T_{\text{BBN}} \approx 1 \text{ MeV}$, then a significant impact on the primordial light-element abundances would have to be expected.

According to the Friedmann equation the expansion rate is given by $H^2 = (8\pi/3) \rho/m_{\text{Pl}}^2$, where ρ is the energy density of the Universe at a given epoch and $m_{\text{Pl}} = 1.22 \times 10^{19} \text{ GeV}$ is the Planck mass. In the radiation-dominated epoch, the energy density is $\rho = g(\pi^2/30) T^4$, with g the effective number of thermally excited degrees of freedom. Between the QCD and BBN epochs we need to count photons,

e^\pm , and the l.h. sequential (anti)neutrinos so that $g = 43/4$. Therefore, we need to require that at some epoch Γ_{depol} exceeds

$$H = f_H (43\pi^3/45)^{1/2} T^2/m_{\text{Pl}} \quad , \quad (24)$$

where $f_H \equiv (g/43)^{1/2}$. The energy density of the magnetic field should be added to ρ , but since it is at most of the same order of magnitude as the radiation energy it can be absorbed into f_H without changing our analysis significantly.

In order to perform this comparison we need to understand the scaling law of an assumed magnetic field distribution under the cosmic expansion. Flux conservation indicates that $B \propto R^{-2}$ or $B \propto T^2$. This would be the complete scaling if the magnetic field were homogeneous. In practice, any primordial field distribution is expected to be complicated and noisy, so that the effective B_{T}^2 in Eq. (23) must be interpreted as a suitable average. Essentially, each neutrino oscillation process “measures” the magnetic field linearly averaged over distances corresponding to the oscillation length so that we need to estimate $\langle \mathbf{B}_{\text{T}} \rangle_{L_{\text{osc}}} \equiv \frac{1}{L_{\text{osc}}} \int_{L_{\text{osc}}} |dx| \mathbf{B}_{\text{T}}(x)$, where the integral is over a neutrino oscillation path. This linear average is relevant because the spin-precession equation (Eq. (14)) is linear in \mathbf{B}_{T} . Further, one must take an ensemble average over all oscillation paths at a given epoch. Here, the average should be taken over the quantity $\langle \mathbf{B}_{\text{T}} \rangle_{L_{\text{osc}}}^2$ since the expression for the depolarization rate in Eq. (23) is quadratic in B_{T} . The co-moving oscillation length L_{osc} increases with time so that at later times the effective field strength is averaged over larger co-moving domains, reducing the effective field strength below the naive T^2 scaling law.

As a simple model for the B -field scaling we assume a power law in co-moving coordinates of the form [10]

$$\langle \langle B_{\text{T}} \rangle \rangle_L \equiv \langle \langle \mathbf{B}_{\text{T}} \rangle_L^2 \rangle^{1/2} = B_d \left(\frac{T}{T_0} \right)^2 \left(\frac{d}{L} \right)^\gamma \quad , \quad (25)$$

where T_0 is the temperature at a reference epoch which we take to be the BBN time with $T_0 = 1 \text{ MeV}$, d is the co-moving size of a typical domain over which the field is correlated, and B_d is the field strength in such a domain. The average $\langle \langle \dots \rangle \rangle_L$ indicates a linear averaging over the length scale L and a root-mean-square average over all oscillation paths. The exact B -field scaling depends on the mechanism of initial generation and the evolution of the complicated magnetohydrodynamic equations [26]. Therefore, the power law in Eq. (25) should only be taken as a toy model for $L \gg d$. In particular, the merging of two domains would change both d and B_d while we take the combination $B_d T_0^{-2} d^\gamma$ to be constant here.

Because oscillating neutrinos measure the magnetic field over an oscillation length scale, it is natural to use $L = L_{\text{osc}}$, which is of order H^{-1} at BBN. Therefore, we define a horizon-scale magnetic field at BBN by $B_0 \equiv \langle \langle B_{\text{T}} \rangle \rangle_{L_{\text{osc}}}$ taken at $T = T_0$. The condition $\Gamma_{\text{depol}} \gtrsim H$ for which r.h. neutrinos are certain to reach thermal equilibrium at some epoch T then translates into

$$\mu B_0 \gtrsim \left(\frac{f_H f_{\text{L}}^2}{f_{\text{D}}} \right)^{1/2} \frac{\pi}{20} \left(\frac{2107 \pi}{45} \right)^{1/4} \frac{G_{\text{F}} T^{3/2} T_0^2}{\alpha m_{\text{Pl}}^{1/2}} \left(\frac{L_{\text{osc}}(T)}{L_{\text{osc}}(T_0)} \right)^\gamma \quad . \quad (26)$$

The temperature dependence of the co-moving oscillation length can be determined from $|\mathbf{V}|^{-1} \lesssim V_L^{-1}$ in Eq. (20) so that $L_{\text{osc}}(T)/L_{\text{osc}}(T_0) = (T_0/T)^4$. If we focus on the period between the QCD phase transition and BBN we have $f_H = 1$, and the numerical coefficient in Eq. (26) is $0.55 \approx 1/2$. Then

$$\mu B_0 \gtrsim \frac{G_F}{2\alpha m_{\text{Pl}}^{1/2}} T^{3/2-4\gamma} T_0^{2+4\gamma} \quad , \quad (27)$$

where in addition we have used $f_D = f_L = 1$ appropriate for ν_μ and ν_τ .

Evidently we need to distinguish between two generic cases depending on whether $\gamma < 3/8$ or $\gamma > 3/8$. Beginning with the former, which would be applicable for a homogeneous field, the condition in Eq. (27) should be imposed at as low a temperature as possible in order to find the smallest necessary μB_0 sufficient to populate the r.h. states. Our entire discussion makes sense only as long as neutrinos scatter efficiently by weak interactions. At later times they may still spin-precess in the cosmic magnetic field, but this would have no further impact on the expansion rate of the Universe as the only effect would be to redistribute the total neutrino energy density between r.h. and l.h. states. Neutrinos freeze out at about $T = 1 \text{ MeV}$, just before the BBN epoch. With $T = T_0 = 1 \text{ MeV}$ r.h. neutrinos reach thermal equilibrium before BBN if

$$\mu B_0 \gtrsim \frac{G_F T_0^{7/2}}{2\alpha m_{\text{Pl}}^{1/2}} \approx 7 \times 10^{-15} \text{ eV} \approx 1.2 \times 10^{-6} \mu_B \text{ gauss} \quad . \quad (28)$$

This requirement is essentially identical to what was found in Ref. [3], even though the interplay between oscillations and collisions was not treated there. It was demanded that the mixing angle should be large, and that the damping rate should be small compared with the spin-precession rate, conditions which are sufficient, but not necessary to achieve thermal equilibrium. Here we found that we are always in the weak-damping case. If we take damping effects into account according to Eq. (14), the required magnitude for μB_0 at the critical epoch around neutrino freeze-out implies that the mixing angle is not small. Therefore, either treatment leads to roughly the same answer. The underlying reason for this coincidence is that in the early Universe the dispersive and the absorptive parts of the neutrino refractive index are of the same general magnitude, i.e. they are both second order in G_F . Then, at the critical epoch around neutrino freeze-out, the time scales $\langle V_L \rangle$, $\langle D \rangle$, and H are all about the same to within numerical factors.

The assumption that the magnetic field is a slowly varying function ($\gamma \approx 0$) on the scale of the Hubble radius at nucleosynthesis is not physically very likely. In fact, an ubiquitous *mean field* would be incompatible with the observed cosmic isotropy if its present strength were larger than about 10^{-7} gauss [24]. Furthermore, it is a general feature of models predicting magnetic field generation during primordial phase transitions [25] to forecast random magnetic fields at the end of the transition, in domains having a typical size $d \ll H^{-1}$. Although magnetohydrodynamical [26] and dissipative effects [27] can cause the ratio d/H^{-1} to grow during the cosmic expansion it may still be much smaller than unity at the BBN time. At that epoch,

the neutrino oscillation length is not much smaller than H^{-1} , so that the neutrino probes a number of field inversions before one spin precession is complete.

If the magnetic field performs a random walk along each neutrino trajectory, the average transverse field decreases with the square root of the length scale. Therefore, one would expect that $\gamma = 1/2$, whence it appears more natural that $\gamma > 3/8$.

In this second generic case the condition Eq. (27) is easiest to fulfil at early times. Typically, the earliest useful epoch is just after the QCD phase transition at $T \approx 150$ MeV. Then, because $\gamma > 3/8$ by assumption, the required value for μB_0 will be smaller than Eq. (28) by a factor $(T_0/T)^{4\gamma-3/2}$, which for $\gamma = 1/2$ is an order of magnitude. Therefore, Eq. (28) is a conservative requirement in the sense that this value for μB_0 is certainly sufficient to populate r.h. neutrinos before BBN, but a smaller value may suffice, depending on the exact scaling law of the effective B -field.

In our derivation we have assumed that the effective mixing angle is small, a condition that we now need to verify. From Eq. (17) we need to demand that $2\langle\mu B_T\rangle/\langle V_L\rangle \lesssim 1$ or

$$\mu B_0 \lesssim f_L \frac{7\pi}{80\alpha} G_F^2 T^{3-4\gamma} T_0^{2+4\gamma} . \quad (29)$$

This condition is most difficult to fulfil at late times, unless $\gamma > 3/4$. Therefore, it is enough to check it at $T = T_0 = 1$ MeV. At that temperature it amounts to $\mu B_0 \lesssim 5 \times 10^{-15}$ eV. A comparison with Eqs. (28) and (29) reveals that our assumption of a small mixing angle has been marginally consistent for $\gamma < 3/8$, and safe for $3/8 < \gamma < 3/4$. Assuming that the mixing angle is large amounts to ignoring refractive effects. This leads to a requirement similar to Eq. (28) for r.h. neutrinos to reach thermal equilibrium.

The magnetically induced spin-oscillation of neutrinos in the early Universe has been discussed in several recent papers [9]. While some of them discuss the importance of neutrino refractive effects at length, this effect does not always matter in their final result. The difference to our treatment is that we study the effect of correlated domains with finite sizes while these papers use the limit where the fields in different points are uncorrelated, $\langle B_i(\mathbf{x}) B_j(\mathbf{y}) \rangle \sim \delta_{ij} \delta^{(3)}(\mathbf{x} - \mathbf{y})$. In that limit, the magnetic field is assumed to consist of very small domains of random magnetic field strength and direction. Therefore, the main difference between our discussion and that of Refs. [9] consists of the assumptions about the magnetic field distribution, and the kinetic treatment adequate for those assumptions.

IV. DISCUSSION AND SUMMARY

We have studied magnetically induced spin precessions of Dirac neutrinos in the early Universe. To this end we have derived expressions for the neutrino dispersion relations in magnetized media which are valid for field strengths B up to about m_W^2 . In the weak-field limit, our results agree with those of D’Olivo, Nieves, and Pal [13] apart from an overall sign. In a charge-symmetric plasma, there is no magnetization contribution to the neutrino refractive index to lowest order in m_W^{-2} , contrary to the

claim of Semikoz and Valle [8]. Besides a formal derivation, we have shown how to obtain the magnetic refraction term in a direct and simple physical fashion, which establishes without ambiguity the absolute sign, and the relative sign between the electron and positron contributions.

Our analysis indicates that r.h. Dirac neutrinos would be thermally populated by spin oscillations if $\mu B_0 \gtrsim 10^{-6} \mu_B$ gauss, where μ is the assumed neutrino magnetic dipole moment, $\mu_B = e/2m_e$ is the Bohr magneton, and B_0 a horizon-scale magnetic field at $T_0 = 1$ MeV, i.e. just before the epoch of nucleosynthesis. Depending on the spatial magnetic-field distribution on smaller scales, i.e. with sufficient power in smaller-scale field modes, even a smaller value of μB_0 would suffice to thermalize the r.h. states.

In principle, r.h. neutrinos could also be populated by direct spin-flip collisions on charged particles or from annihilation processes involving virtual photons [28]. The dipole moment needed to achieve thermal equilibrium for the r.h. states is $\mu \gtrsim 0.5 \times 10^{-10} \mu_B$. If the neutrino mass is smaller than 1 MeV, as we assume in the present paper, stellar-evolution bounds on neutrino dipole or transition moments are $\mu \lesssim 3 \times 10^{-12} \mu_B$ [1,29], so that the scattering mechanism cannot be effective in the early Universe.

In the particle-physics standard model, neutrinos have no magnetic dipole moments. However, if neutrinos have a Dirac mass m they automatically have a dipole moment $\mu/\mu_B = 3.2 \times 10^{-19} m/\text{eV}$. In other extensions of the standard model much larger values can be obtained. If one of the neutrinos would saturate the stellar-evolution limit, a primordial field at nucleosynthesis $B_0 \approx 3 \times 10^5$ gauss would be enough to populate the r.h. degrees of freedom, and an even smaller field could suffice, depending on its spatial distribution.

Unfortunately, direct observations of primordial magnetic fields are still lacking, although it was recently suggested that they may be detectable by observing their imprint on the cosmic rays [30] or on the cosmic microwave radiation [31]. However, it may be useful to consider some recent hypotheses about the genesis and evolution of primordial magnetic fields. Many of these propositions are motivated by the desire to explain the observed galactic and intergalactic magnetic fields as relics of a primordial cosmological field. Field strengths of order 10^{-6} Gauss are a quite general character of the interstellar medium. Remarkably, this strength corresponds to an energy density equal to that of the cosmic microwave background radiation. Kronberg [32] suggests that this feature may be the result of an early equipartition between magnetic fields and radiation, a hypothesis that may have found some theoretical support (e.g. Ref. [33]). If this were the case we could expect $B_0 \approx 10^{13}$ Gauss, a value which is not in contradiction with primordial nucleosynthesis considerations [34]. If such large fields were produced before nucleosynthesis, our result implies that even a dipole moment as small as about $10^{-19} \mu_B$ would be enough to thermalize r.h. neutrinos. Thus, neutrinos with cosmologically interesting Dirac masses in the eV range would have sufficiently large dipole moments without further extensions of the standard model.

It has frequently been argued that the observationally inferred primordial light-element abundances exclude significant novel contributions to the cosmic expansion

rate of the Universe at the nucleosynthesis epoch. At the present time, however, new questions concerning the reliability of the previously inferred abundances of deuterium and ^3He have arisen, and the overall consistency of BBN with all of the observations is not assured [35]. Therefore, at the present time one cannot assume that the observationally inferred primordial light-element abundances truly exclude one additional thermally excited neutrino degree of freedom at the nucleosynthesis epoch. Therefore, it is not the ambition of our present study to claim a new exclusion range for μB_0 , but rather to illuminate some of the important physical ingredients needed to understand magnetically induced neutrino spin oscillations in the early Universe.

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NOTE ADDED

Before circulating the present paper as an E-print we made a draft version available to Drs. J.W.F. Valle and V. Semikoz who subsequently agreed that our expression for the magnetically induced refractive index was the correct one. As a formal response they have now circulated a corrected version of their derivation [36] which explicitly confirms our finding.

APPENDIX A: CHARGED-FERMION PROPAGATOR

In order to calculate the tadpole and bubble diagrams in Sect. II, we need an explicit expression for the electron propagator $S(x', x'')$ in the presence of an external magnetic field and an electronic plasma at non-zero temperature and density. We shall use two different methods here: Furry's picture for the local term and Schwinger's proper-time method for the non-local one. They give the same result for the local terms, but the Furry-picture result is more direct to interpret physically. For the non-local terms it would be considerably more difficult to use the Furry picture.

By the Furry picture we mean that the propagator is constructed explicitly as a sum over solutions to the Dirac equation in a given gauge. For a fermion with mass m and charge q (the electron having a negative charge $q = -e < 0$) the propagator

has been constructed in Refs. [22,23]. For a magnetic field in the positive z -direction, in the gauge $A_\mu = (0, 0, -Bx, 0)$, it is given by

$$iS(x, x) = \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dp_y}{2\pi} \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} \left[\frac{i}{p_0^2 - E_{l,p_z}^2} - 2\pi\delta(p_0^2 - E_{l,p_z}^2) f_F(p_0) \right] \\ \times \left\{ (p_0\gamma_0 - p_z\gamma_z + m) [\sigma_+ I_{l,l}(x, p_y) + \sigma_- I_{l-1,l-1}(x, p_y)] \right. \\ \left. - i\sqrt{2l|qB|} [\gamma_+ I_{l,l-1}(x, p_y) - \gamma_- I_{l-1,l}(x, p_y)] \right\} , \quad (\text{A1})$$

where² $\gamma_\pm \equiv \frac{1}{2}[\gamma_x \pm \text{sign}(qB)i\gamma_y]$ and $\sigma_\pm \equiv \frac{1}{2}[1 \pm \text{sign}(qB)\sigma_z]$. Note that σ_z is understood to mean the Dirac spin matrix $\frac{i}{2}[\gamma_x, \gamma_y]$. The Landau levels are labelled by l and their energies are $E_{l,p_z}^2 = m^2 + p_z^2 + 2|qB|l$. Further, $f_F(p_0) = f_F^+(p_0)\Theta(p_0) + f_F^-(p_0)\Theta(-p_0)$, where $f_F^\pm(p_0) = (e^{\pm(p_0 - \mu)/T} + 1)^{-1}$ are the usual occupation numbers for particles and antiparticles of a Fermi-Dirac distribution at temperature T and chemical potential μ . We have also used $I_{k,l}(x, p_y) \equiv I_k(x, p_y)I_l(x, p_y)$ with

$$I_l(x, p_y) = \left(\frac{|qB|}{\pi} \right)^{1/4} \exp \left[-\frac{|qB|}{2} \left(x - \frac{p_y}{qB} \right)^2 \right] \frac{1}{\sqrt{l!}} H_l \left[\sqrt{2|qB|} \left(x - \frac{p_y}{qB} \right) \right] , \quad (\text{A2})$$

where H_l is a Hermite polynomial. In the lowest Landau level we define $I_{-1} = 0$ for consistency.

The dp_y integration can be performed by using the completeness relation

$$\int_{-\infty}^{+\infty} dp_y I_k(x, p_y) I_l(x, p_y) = |qB| \delta_{kl} , \quad (\text{A3})$$

which also removes the x -dependence from the r.h.s. of Eq. (A1). In the end we are only interested in the thermal part, coming from the δ -function in Eq. (A1), so we drop the vacuum contribution from now on. After the dp_z integration has been done using the δ -function we find

$$iS(x, x) = -\frac{|qB|}{4\pi^2} \int_{-\infty}^{+\infty} dp_0 f_F(p_0) (\gamma_0 p_0 + m) \\ \times \left(\frac{\Theta(p_0^2 - m^2)}{\sqrt{p_0^2 - m^2}} \sigma_+ + \sum_{l=1}^{\infty} \frac{\Theta(p_0^2 - m^2 - 2|qB|l)}{\sqrt{p_0^2 - m^2 - 2|qB|l}} \right) . \quad (\text{A4})$$

The appearance of the projection operator σ_+ in Eq. (A4) is related to the fact that there is only one possible spin orientation in the lowest Landau level ($l = 0$).

With this result it is straightforward to calculate expectation values like

²In our convention three-vectors such as $\mathbf{p} = (p_x, p_y, p_z)$ and $\boldsymbol{\gamma} = (\gamma_x, \gamma_y, \gamma_z)$ are the contravariant components of the corresponding four-vector and thus have Lorentz indices $i = 1, 2, 3$ upstairs, i.e. $p_x = p^1$ etc. We use the Minkowski metric $\text{diag}(+, -, -, -)$ so that $p_i = -p^i$ and $\gamma_i = -\gamma^i$ for $i = 1, 2, 3$.

$$\langle \bar{\Psi}(x) \gamma^i \gamma_5 \Psi(x) \rangle = -\text{tr} [iS(x, x) \gamma^i \gamma_5] \quad , \quad (\text{A5})$$

where the trace is over γ -matrices. Since $\gamma_i \gamma_5$ contains an odd number of γ -matrices, only the term in the integrand in Eq. (A4), which is odd in p_0 , can contribute. Evidently it is zero for a vanishing chemical potential, showing in a more formal way that the magnetization term of Semikoz and Valle [8] cannot be correct.

It is often more convenient to label the Landau levels with an orbital quantum number $n = 0, 1, 2, \dots$ and a spin quantum number $\lambda = \pm 1$. The energies are then $E_{n,\lambda,p_z}^2 = m^2 + p_z^2 + |qB|(2n + 1 - \lambda)$. For a charged Dirac fermion f the net total number density (particles minus antiparticles) is

$$N_{f-\bar{f}} = \frac{|qB|}{2\pi^2} \int_0^\infty dp_z \sum_{n=0}^\infty \sum_{\lambda=\pm 1} \left[f_F^+(E_{n,\lambda,p_z}) - f_F^-(E_{n,\lambda,p_z}) \right] \quad , \quad (\text{A6})$$

while the net number density in the lowest Landau level is

$$N_{f-\bar{f}}^0 = \frac{|qB|}{2\pi^2} \int_0^\infty dp_z \left[f_F^+(E_{0,1,p_z}) - f_F^-(E_{0,1,p_z}) \right] \quad . \quad (\text{A7})$$

These results allow us to relate the local terms of the neutrino self-energy to the total charge density or to the charge density in the lowest Landau level, leading to Eqs. (4) and (5).

For the non-local neutrino self-energy term it is convenient to start from the electron propagator in the Schwinger proper-time form, which can be written as [19,21]:

$$iS(x', x'') = \phi(x', x'') \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x'-x'')} iS(p) \quad , \quad (\text{A8})$$

where $\phi(x', x'')$ is a gauge-dependent phase factor. The gauge-independent and translationally invariant part of S is

$$iS(p) = iS_{\text{vac}}(p) - f_F(p_0) \left[iS_{\text{vac}}(p) - iS_{\text{vac}}^*(p) \right] \quad , \quad (\text{A9})$$

where

$$\begin{aligned} iS_{\text{vac}}(p) = & \int_0^\infty ds \frac{e^{iqBs\sigma_z}}{\cos(qBs)} \exp \left[is \left(p_\parallel^2 - \frac{\tan(qBs)}{qBs} p_\perp^2 - m^2 + i\varepsilon \right) \right] \\ & \times \left(\gamma p_\parallel - \frac{e^{-iqBs\sigma_z}}{\cos(qBs)} \gamma p_\perp + m \right) \quad , \end{aligned} \quad (\text{A10})$$

where for general four-vectors a and b , $a \cdot b_\parallel = a_0 b_0 - (\hat{\mathbf{B}} \cdot \mathbf{a})(\hat{\mathbf{B}} \cdot \mathbf{b})$ and $a \cdot b_\perp = \mathbf{a} \cdot \mathbf{b} - (\hat{\mathbf{B}} \cdot \mathbf{a})(\hat{\mathbf{B}} \cdot \mathbf{b})$. The real combination that occurs in the thermal part of Eq. (A9) is obtained by extending the s -integral in Eq. (A10) from $-\infty$ to $+\infty$. In the integrand of Eq. (A10) there are poles and essential singularities on the real s -axis. They have to be avoided by taking the integration contour in the lower half-plane for positive s (see e.g. Ref. [17] for a discussion of this contour). Therefore, to

get a real quantity for the thermal part, this contour has to go in the lower half-plane for negative s as well.

The W boson propagator has a similar form but with a different tensor structure. In a closed loop, the gauge-dependent phase factors $\phi(x, x')$ cancel and the result is explicitly translationally invariant. The contribution from thermal W bosons is Boltzmann, suppressed by a factor $e^{-m_W/T}$ and can be neglected. Expanding the W propagator in both the momentum transfer and the B field we find that the leading B -dependent $\mathcal{O}(m_W^{-4})$ -term is local and that the first non-local B -dependent term is $\mathcal{O}(m_W^{-6})$. The local term vanishes in a CP symmetric plasma. Therefore, when calculating the neutrino self-energy to order m_W^{-4} we may use the zero-field W propagator.

The advantage with the Schwinger proper-time form over the Furry picture is that the gauge-dependent phase factor disappears automatically and we do not have to match the wave functions of the electron propagator (i.e. the Landau levels) with the ones of the W propagator in the zero field limit (i.e. plane waves).

With the propagators in Eqs. (3) and (A9) it is possible to perform the loop integral over the three-momenta in Eq. (2) explicitly, but the result is still fairly complicated. It simplifies considerably in the linear-field approximation ($B \ll T^2$, $B \ll m^2$), where we have, from the W - e -loop:

$$\begin{aligned} \Sigma_{\text{bubble}} = & -\frac{g^2}{2m_W^4} \int \frac{d^4 p}{(2\pi)^4} f_F(p_0) \int_{-\infty}^{\infty} ds e^{is(p^2 - m^2)} \gamma^\mu \left[\gamma p + m + iq \hat{\mathbf{B}} \cdot \boldsymbol{\sigma} s(\gamma p_{\parallel} + m) \right] \\ & \times \left[g_{\mu\nu}(k - p)^2 - (k - p)_\mu (k - p)_\nu \right] \gamma^\nu \quad . \end{aligned} \tag{A11}$$

After adding the Z - ν -loop and keeping only the leading high-temperature piece, we obtain the result in Eq. (12). However, Eq. (A11) is valid also for temperatures lower than the electron mass m . It contains corrections to Eq. (12), which can be important if $T \lesssim m$.

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